CONTINUED FRACTIONS AND TOTALLY MONOTONE SEQUENCES

BY

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1. **Introduction.** A sequence $c_0, c_1, c_2, \cdots$ of real numbers is called *totally monotone* if $\Delta^m c_n \geqq 0$, $(m, n = 0, 1, 2, \cdots)$, where

$$\Delta^m c_n = c_n - C_{m,1} c_{n+1} + C_{m,2} c_{n+2} - \cdots + (-1)^m C_{m,m} c_{n+m}.$$

Hausdorff[1] showed that for every totally monotone sequence $c_0, c_1, c_2, \cdots$ there exists (essentially uniquely) a monotone nondecreasing real function $\phi(u)$, $0 \leqq u \leqq 1$, such that

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots.$$

Conversely, if $\phi(u)$ is a monotone nondecreasing bounded real function on the interval $0 \leqq u \leqq 1$, then $\Delta^m c_n = \int_0^1 (1-u)^m u^n d\phi(u) \geqq 0$, $m, n = 0, 1, 2, \cdots$, so that $c_0, c_1, c_2, \cdots$ is totally monotone.

In case the function $\phi(u)$ has an infinity of points of increase in the interval $0 \leqq u \leqq 1$, then the corresponding sequence is a special *Stieltjes moment sequence*, and accordingly there is a Stieltjes continued fraction[2]

$$(1.1) \qquad b_1/1 + b_2 x/1 + b_3 x/1 + \cdots,$$

in which the numbers $b_1, b_2, b_3, \cdots$ are real and positive, which corresponds to the power series

$$(1.2) \qquad c_0 - c_1 x + c_2 x^2 - \cdots.$$

On the other hand, if $\phi(u)$ has but a finite number of points of increase, then the series (1.2) represents a rational function of x and the continued fraction terminates.

The main problem which we have solved in the present paper is as follows: *to find necessary and sufficient conditions upon the numbers $b_1, b_2, b_3, \cdots$ in the continued fraction* (1.1) *in order that the coefficients $c_0, c_1, c_2, \cdots$ in the corresponding power series* (1.2) *shall form a totally monotone sequence.* The re-

sult is very simple, namely: the sequence c_0, c_1, c_2, $\cdots$ is totally monotone if and only if there exist real numbers g_0, g_1, g_2, $\cdots$ such that $0 \leqq g_n \leqq 1$, $n = 0, 1, 2, \cdots$, and such that

$$c_0 - c_1 x + c_2 x^2 - c_3 x^3 + \cdots \sim g_0/1 + g_1 x/1 + (1 - g_1)g_2 x/1$$
$$+ (1 - g_2)g_3 x/1 + \cdots ,$$

it being agreed that the continued fraction[3] shall terminate with the first identically vanishing partial quotient.

If c_0, c_1, c_2, $\cdots$ is totally monotone, the function $f(x)$ represented by the power series (1.2) is analytic for $|x| < 1$. Let $M(f) = \text{l.u.b.}_{|x|<1} |f(x)|$. We show that $M(f)$ is finite if and only if the series $c_0 + c_1 + c_2 + \cdots$ converges, and establish the equality

$$M(f) = c_0 + c_1 + c_2 + \cdots .$$

We also characterize the class E of these "moment generating functions" which are bounded in the unit circle (a) in terms of the Stieltjes integral representation of $f(x)$, and (b) in terms of the continued fraction representation of $f(x)$. It is shown that if $f(x) \ \varepsilon \ E$, $M(f) \leqq 1$, then the functions defined by the algorithm of Schur[4], namely:

$$f_{n+1} = \frac{1}{x} \frac{t_n - f_n}{1 - t_n f_n}, \qquad t_n = f_n(0),$$

$n = 0, 1, 2, \cdots$, $f_0 = f$, are all in E and have moduli not exceeding 1 for $|x| < 1$.

2. **An operation on continued fractions.** We shall use the symbol "$\sim$" between a power series $P(x)$ and a continued fraction $K(x)$ to indicate that the power series expansion of the nth approximant of $K(x)$ agrees term by term with $P(x)$ for more and more terms as n is increased, or becomes identical with $P(x)$ from and after some value of n. The basic theorem of the paper is

THEOREM 2.1. *If g_1, g_2, g_3, $\cdots$ are any real or complex numbers, and $P(x)$ is a power series in ascending powers of x such that*

$$(2.1) \qquad P(x) \sim 1 + g_1 x/1 + (1 - g_1)g_2 x/1 + (1 - g_2)g_3 x/1 + \cdots ,$$

then

$$(2.2) \qquad \frac{1 + x}{P(x)} \sim 1 + (1 - g_1)x/1 + g_1(1 - g_2)x/1 + g_2(1 - g_3)x/1 + \cdots .$$

[3] Continued fractions of this form were first treated by E. B. Van Vleck, in a paper entitled *On the convergence and character of the continued fraction $a_1 z/1 + a_2 z/1 + a_3 z/1 + \cdots$*, these Transactions, vol. 2 (1901), pp. 476–483.

[4] J. Schur, *Ueber Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 205–232, and vol. 148 (1917), pp. 122–145.

Proof. Let $A_n(x)/B_n(x)$, $A_n^*(x)/B_n^*(x)$ be the nth approximants of the continued fractions in (2.1) and (2.2), respectively. Then we have the relations

$$(2.3) \quad A_n(x) = g_n x B_{n-1}^*(x) + B_n^*(x), \quad (1 + x)B_n(x) = g_n x A_{n-1}^*(x) + A_n^*(x),$$

($n = 0, 1, 2, \cdots$, $g_0 = 1$, $A_{-1}^* = 1$, $B_{-1}^* = 0$). These may be verified directly for $n = 0$, 1. Assuming that the first is true for $n \leqq m$, $m \geqq 1$, we then have

$$A_{m+1}(x) = A_m(x) + g_{m+1}(1 - g_m)x A_{m-1}(x)$$
$$= g_m x B_{m-1}^*(x) + B_m^*(x) + g_{m+1}(1 - g_m)x[g_{m-1}x B_{m-2}^*(x) + B_{m-1}^*(x)].$$

Since $g_{m-1}(1 - g_m)x B_{m-2}^*(x) = B_m^*(x) - B_{m-1}^*(x)$, we then have

$$A_{m+1}(x) = g_{m+1} x B_m^*(x) + [B_m^*(x) + g_m(1 - g_{m+1})x B_{m-1}^*(x)]$$
$$= g_{m+1} x B_m^*(x) + B_{m+1}^*(x),$$

so that the first relation (2.3) is true for $n = m+1$, and therefore, by mathematical induction, for all n. The second relation (2.3) may be proved in a similar way.

On multiplying the first relation (2.3) by $A_{n-1}^*(x)$, the second by $B_{n-1}^*(x)$, and then subtracting, we find that the power series expansion in ascending powers of x of the rational function

$$A_{n-1}^*(x)/B_{n-1}^*(x) - \frac{1 + x}{A_n(x)/B_n(x)}$$

begins with the $(n-1)$th or a higher power of x. It follows immediately that the correspondence (2.1) implies the correspondence (2.2).

This theorem may be thrown into the following form:

THEOREM 2.2. *Let $c_0 \neq 0$, and*

$$(2.4) \quad c_0 - c_1 x + c_2 x^2 - \cdots \sim c_0/1 + g_1 x/1 + (1 - g_1)g_2 x/1$$
$$+ (1 - g_2)g_3 x/1 + \cdots .$$

Then

$$(2.5) \quad \Delta c_0 - \Delta c_1 x + \Delta c_2 x^2 - \cdots \sim \Delta c_0/1 + g_1(1 - g_2)x/1$$
$$+ g_2(1 - g_3)x/1 + \cdots .$$

Proof. Let $c_0/P(x) = c_0 - c_1 x + c_2 x^2 - \cdots$ in Theorem 2.1. This gives at once

$$c_0 + (c_0 - c_1)x - (c_1 - c_2)x^2 + \cdots \sim c_0 + c_0(1 - g_1)x/1$$
$$+ g_1(1 - g_2)x/1 + \cdots .$$

On removing the constant term c_0 from the series and from the continued fraction, and then dropping a factor x, the correspondence (2.5) results. We note for future reference that

(2.6) $$\Delta c_0 = c_0(1 - g_1).$$

By means of these theorems we have enlarged by one the small list of known operations on continued fractions.

The next theorem makes the transformation available for a large class of continued fractions.

THEOREM 2.3. *Let $A_n(x)/B_n(x)$ be the nth approximant of the continued fraction*

(2.7) $$1 + a_1x/1 + a_2x/1 + a_3x/1 + \cdots ,$$

and let $c \neq 0$ be any number such that $A_n(-c) \neq 0$, $(n = 1, 2, 3, \cdots)$. Put

(2.8) $$g_n = a_n c A_{n-2}(-c)/A_{n-1}(-c),$$

$n = 1, 2, 3, \cdots , A_{-1}(-c) = 1$. Then the continued fraction (2.7) takes the form

$$1 + \frac{g_1(x/c)}{1} + \frac{(1 - g_1)g_2(x/c)}{1} + \frac{(1 - g_2)g_3(x/c)}{1} + \cdots .$$

Proof. One may verify immediately that $a_1 = g_1/c$, $a_n = g_n(1 - g_{n-1})/c$, $n = 2, 3, 4, \cdots$, when the g_n's are given by (2.8).

We remark in passing that if $a_n \neq 0$ in (2.7), and $P(x)$ is the power series corresponding to (2.7), then one may apply Theorem 2.1 to obtain the formulas given by Stieltjes[5] for the continued fraction corresponding to $1/P(x)$. In fact:

(2.9) $$\frac{1 + (x/c)}{P(x)} \sim 1 + \frac{(1 - g_1)(x/c)}{1} + \frac{g_2(1 - g_1)(x/c)}{1}$$
$$+ \frac{g_3(1 - g_2)(x/c)}{1} + \cdots .$$

We may allow c to become infinite and the correspondence will be maintained provided the coefficients of x in the continued fraction have limits which are finite and not 0. Accordingly we obtain this theorem:

THEOREM 2.4. *If the power series $P(x)$ has a corresponding continued fraction (2.7) in which $a_n \neq 0$, $n \geq 1$, then $1/P(x)$ will have a corresponding continued fraction of the same form provided $A_{2n}(x)$, the numerator of the 2nth approximant of (2.7), is of degree n for $n = 0, 1, 2, \cdots$.*

The condition of the theorem is met when the a_n's are real and positive, which is the case with which Stieltjes was concerned. If one evaluates the limits for $c = \infty$ of the coefficients of x in the continued fraction of (2.9), he will find that the result agrees with that found by Stieltjes.

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[5] O. Perron, *Die Lehre von den Kettenbrüchen*, 1st edition, pp. 334–335.

3. A uniform convergence theorem. In a recent([6]) paper Scott and Wall proved a theorem which may be stated in the following form:

THEOREM 3.1. *If g_1, g_2, g_3, $\cdots$ are real numbers such that $0 < g_1 < 1$, $0 \leqq g_n < 1$, $n > 1$, and x_1, x_2, x_3, $\cdots$ are functions of any variables, then the continued fraction*

$$(3.1) \qquad \frac{1}{1} + \frac{(1 - g_1)g_2 x_1}{1} + \frac{(1 - g_2)g_3 x_2}{1} + \frac{(1 - g_3)g_4 x_3}{1} + \cdots$$

converges uniformly for $|x_n| \leqq 1$, $n = 1, 2, 3, \cdots$. The denominators of all the approximants are nonzero in this domain. Let G denote the value of the continued fraction. Then

$$|G| \leqq \frac{1}{g_1} \left\{ 1 - \frac{1}{1 + \sum_1^\infty \dfrac{g_1 g_2 \cdots g_n}{(1 - g_1)(1 - g_2) \cdots (1 - g_n)}} \right\}$$

if $|x_n| \leqq 1$, $n \geqq 1$; and G is equal to the expression on the right if $x_n = -1$, $n \geqq 1$.

We shall digress momentarily at this point to discuss two theorems given by Perron on page 262 of his book. The first of these may, with no essential loss in generality, be stated as follows:

THEOREM 3.2. *If the elements of the continued fraction*

$$(3.2) \qquad 1/1 + a_2/1 + a_3/1 + a_4/1 + \cdots$$

are functions of any variables, then the continued fraction converges uniformly over the domain characterized by the inequalities

$$|a_n| \leqq (p_n - 1)/p_n p_{n-1}, \qquad n = 2, 3, 4, \cdots,$$

where p_1, p_2, p_3, $\cdots$ are any real constants greater than 1 for which the series

$$(3.3) \qquad \sum_1^\infty (p_1 - 1)(p_2 - 1) \cdots (p_n - 1)$$

is divergent.

Perron then says: "Ein bemerkenswerter Spezialfall unseres allgemeinen Kriteriums ist" and then *proves* a theorem, attributed to Van Vleck, which may be stated as follows:

THEOREM 3.3. *If g_1, g_2, g_3, $\cdots$ are real numbers such that $0 < g_n < 1$, $n \geqq 1$, and x_1, x_2, x_3, $\cdots$ are functions of any variables, then the continued fraction (3.1) converges uniformly for $|x_n| \leqq 1$, $n \geqq 1$.*

([6]) W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions*, these Transactions, vol. 47 (1940), pp. 155–172.

It is a strange fact that *the second theorem is more general than the first.*
To see this, put $a_n = g_n(1 - g_{n-1})x_n$, $n \geq 2$; $p_n = 1/(1 - g_n)$, $n \geq 1$, and the second
theorem reduces to the first *minus* the requirement on the series (3.3).

It should be added that Theorem 3.3 is related to but quite different from
the theorem which Van Vleck proved[7]. He gave preference to the continued
fraction $1/1 + g_1x_0/1 + (1 - g_1)g_2x_1/1 + (1 - g_2)g_3x_2/1 + \cdots$, the reciprocal of
which is, except for an unimportant term and factor, the continued fraction
(3.1). Theorem 3.1 is an improvement over Theorem 3.3, in that the g_n's
after the first are permitted to be 0. For this reason Theorem 3.1 contains
the theorem given by Perron on page 258 (Theorem 26), in which it may be
assumed with no loss in generality that $p_1 > 1$.

**4. Totally monotone sequences corresponding to an "infinite distribution
of mass."** The sequence $c_n = \int_0^1 u^n d\phi(u)$, $n = 0, 1, 2, \cdots$, in which $\phi(u)$ is real
and monotone nondecreasing is completely characterized by the inequalities

$$(4.1) \qquad\qquad\qquad \Delta^m c_n \geq 0, \qquad\qquad m, n = 0, 1, 2, \cdots,$$

and is said to be a *totally monotone* sequence. If $\phi(u)$ has an infinite number of
points of increase, we shall say that there is an *infinite distribution of mass.*

THEOREM 4.1. *The sequence $c_0, c_1, c_2, \cdots$ is a totally monotone sequence
corresponding to an infinite distribution of mass if and only if there exist real
numbers $g_1, g_2, g_3, \cdots$ such that $0 < g_n < 1$, $n \geq 1$, and such that*

$$(4.2) \qquad c_0 - c_1x + c_2x^2 - \cdots \sim c_0/1 + g_1x/1 + (1 - g_1)g_2x/1$$
$$+ (1 - g_2)g_3x/1 + \cdots, \qquad (c_0 > 0).$$

Proof. Suppose that (4.2) holds. Then by Theorem 3.1 the continued frac-
tion $1 + g_1x/1 + (1 - g_1)g_2x/1 + (1 - g_2)g_3x/1 + \cdots$ converges uniformly for
$|x| \leq 1$. If $f(x)$ is the analytic function represented, then by Theorem 2.1
and Theorem 3.1, $(1 + x)/f(x)$ is analytic for $|x| < 1$, and therefore $c_0/f(x)$,
the function represented by the continued fraction and series (4.2), is analytic
for $|x| < 1$.

Now the coefficients of x in the continued fraction (4.2) are positive, and
hence by the work of Stieltjes, this continued fraction represents a function
of the form $\int_0^\infty d\phi(u)/(1 + xu)$, where $\phi(u)$ is monotone nondecreasing, and
has an infinite number of points of increase. Inasmuch as this function is
analytic for $|x| < 1$, and the corresponding continued fraction converges uni-
formly for $|x| \leq r$ where r is any positive number less than 1, it follows
that the upper limit of integration may be taken equal to 1. Then
$\Delta^m c_n = \int_0^1 (1 - u)^m u^n d\phi(u)$, and therefore (4.1) holds. Thus the sequence is
totally monotone, and corresponds to an infinite distribution of mass.

Conversely, let $c_n = \int_0^1 u^n d\phi(u)$, where $\phi(u)$ is monotone nondecreasing and
has an infinite number of points of increase. That is, $c_0, c_1, c_2, \cdots$ is a totally

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[7] See footnote 3.

monotone sequence corresponding to an infinite distribution of mass. Then, by the work of Stieltjes, we must have a correspondence of the form

$$c_0 - c_1 x + c_2 x^2 - \cdots \sim a_1/1 + a_2 x/1 + a_3 x/1 + \cdots,$$

where $a_1, a_2, a_3, \cdots$ are real and positive. Moreover, the function represented by this series and continued fraction is $\int_0^1 d\phi(u)/(1+xu)$. Since the limits of integration are from 0 to 1, the zeros[8] of $B_n(x)$, the denominator of the nth approximant of the continued fraction, are real and less than -1. Since $B_n(0) = 1$ it therefore follows that $B_n(-1) > 0$. We may then apply Theorem 2.3, with $c = 1$, to the continued fraction

$$1 + a_2 x/1 + a_3 x/1 + a_4 x/1 + \cdots,$$

the numerator of whose nth approximant is $B_{n+1}(x)$, and thus determine numbers $g_1, g_2, g_3, \cdots$, such that

$$a_2 = g_1, \qquad a_n = g_{n-1}(1 - g_{n-2}), \qquad n > 2.$$

Now, by Theorem 2.2,

$$\Delta c_0 - \Delta c_1 x + \Delta c_2 x^2 - \cdots \sim \frac{\Delta c_0}{1 +} \frac{g_1(1 - g_2)x}{1 +} \frac{g_2(1 - g_3)x}{1 +} \cdots.$$

Also, $\Delta c_n = \int_0^1 u^n d\phi_1(u)$ where $\phi_1(u) = \int_0^u (1 - u)d\phi(u)$ is monotone nondecreasing and has an infinite number of points of increase. It follows that the coefficients of x in the last continued fraction must all be positive, and that $\Delta c_0 = c_0(1 - g_1) > 0$ (cf. (2.6)). We therefore have

$$g_1 > 0, \qquad g_{n-1}(1 - g_{n-2}) > 0, \qquad n > 2,$$
$$1 - g_1 > 0, \qquad g_{n-2}(1 - g_{n-1}) > 0, \qquad n > 2,$$

and consequently $0 < g_n < 1$, $n = 1, 2, 3, \cdots$, as was to be proved.

5. **Developments from the continued fraction algorithm.** We shall begin by considering an example. It is known[9] that the series

$$F(\alpha, 1, \gamma; -x) = 1 - \frac{\alpha}{\gamma} x + \frac{\alpha(\alpha + 1)}{\gamma(\gamma + 1)} x^2 - \cdots$$

has the corresponding continued fraction $1 + e_1 x/1 + e_2 x/1 + e_3 x/1 + \cdots$, where

$$e_1 = \frac{\alpha}{\gamma}, \qquad e_{2n} = \frac{(\alpha + n)(\gamma + n - 1)}{(\gamma + 2n - 2)(\gamma + 2n - 1)}, \qquad e_{2n+1} = \frac{n(\gamma + \alpha + n - 1)}{(\gamma + 2n - 1)(\gamma + 2n)}.$$

Naturally α, γ are not negative integers or 0. We then readily find that

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[8] Perron, loc. cit., p. 368 and p. 383.
[9] Perron, loc. cit., p. 348.

$$F(\alpha, 1, \gamma; -x) = \cfrac{1}{1} + \cfrac{g_1 x}{1} + \cfrac{(1-g_1)g_2 x}{1} + \cfrac{(1-g_2)g_3 x}{1} + \cdots,$$

where $g_{2n} = n/(\gamma+2n-1)$, $g_{2n-1} = (\alpha+n-1)/(\gamma+2n-2)$, $n = 1, 2, 3, \cdots$. On applying Theorem 2.1 we readily obtain the power series identity

$$(5.1) \quad F(\alpha, 1, \gamma; -x) = \frac{1}{1+x} + \frac{\gamma-\alpha}{\gamma} \frac{x}{1+x} F(\alpha, 1, \gamma+1; -x).$$

The repeated application of this identity gives the Euler expansion

$$F(\alpha, 1, \gamma; -x) = \frac{1}{1+x} + \frac{(\gamma-\alpha)}{\gamma} \frac{x}{(1+x)^2}$$
$$+ \frac{(\gamma-\alpha)(\gamma+1-\alpha)}{\gamma(\gamma+1)} \frac{x^2}{(1+x)^3} + \cdots.$$

We now propose to obtain the analogous developments for the general continued fraction of this form. For simplicity, let g_n be real and $0 < g_n < 1$, and put

$$f(x) = 1 + g_1 x/1 + (1-g_1)g_2 x/1 + (1-g_2)g_3 x/1 + \cdots,$$

so that

$$(1+x)/f(x) = 1 + (1-g_1)x/1 + g_1(1-g_2)x/1 + g_2(1-g_3)x/1 + \cdots.$$

Let $f_1(x) = 1 + g_1(1-g_2)x/1 + g_2(1-g_3)x/1 + \cdots$, and denote by $A_n(x)$ the numerator of the nth approximant of this continued fraction. Then it is easy to prove by mathematical induction that

$$A_{n-1}(-1) = g_1 g_2 \cdots g_n S_n,$$

where

$$S_n = 1 + \sum_{k=1}^{n} \frac{(1-g_1)(1-g_2) \cdots (1-g_k)}{g_1 g_2 \cdots g_k}.$$

We may therefore apply Theorem 2.3 with $c = 1$ and obtain

$$f_1(x) = 1 + g_1^{(1)} x/1 + g_2^{(1)}(1-g_1^{(1)})x/1 + g_3^{(1)}(1-g_2^{(1)})x/1 + \cdots,$$

where $g_n^{(1)} = g_n(1-g_{n+1})A_{n-2}(-1)/A_{n-1}(-1) = (1-g_{n+1})S_{n-1}/S_n$, $n = 1, 2, 3, \cdots$, $(S_0 = 1)$. *Consequently* $0 < g_n^{(1)} < 1$, *so that the continued fraction for $f_1(x)$ has precisely the same form as that for $f(x)$.* Hence we may write

$$\frac{1+x}{f_1(x)} = 1 + \cfrac{(1-g_1^{(1)})x}{1} + \cfrac{g_1^{(1)}(1-g_2^{(1)})x}{1} + \cfrac{g_2^{(1)}(1-g_3^{(1)})x}{1} + \cdots,$$

put $f_2(x) = 1 + g_1^{(1)}(1-g_2^{(1)})x/1 + g_2^{(1)}(1-g_3^{(1)})x/1 + \cdots$, and apply the above argument to the latter continued fraction. In this manner we arrive at the following theorem:

THEOREM 5.1. *Let* $f(x) = 1 + g_1 x/1 + (1-g_1)g_2 x/1 + (1-g_2)g_3 x/1 + \cdots$, *where* $0 < g_n < 1$. *Define numbers* $g_n^{(m)}$ *by the relations*

$$(5.2) \qquad g_k^{(0)} = g_k, \qquad g_k^{(m)} = (1 - g_{k+1}^{(m-1)})S_{k-1}^{(m-1)}/S_k^{(m-1)},$$

$m, k = 1, 2, 3, \cdots,$

$$(5.3) \quad S_0^{(k)} = 1, \quad S_i^{(k)} = 1 + \sum_{j=1}^{i} \frac{(1 - g_1^{(k-1)})(1 - g_2^{(k-1)}) \cdots (1 - g_j^{(k-1)})}{g_1^{(k-1)} g_2^{(k-1)} \cdots g_j^{(k-1)}},$$

so that $0 < g_n^{(m)} < 1$; *and put* $f_k(x) = 1 + g_1^{(k)}x/1 + (1-g_1^{(k)})g_2^{(k)}x/1 + (1-g_2^{(k)})g_3^{(k)}x/1 + \cdots$. *Then the functions* $f_0 \equiv f, f_1, f_2, f_3, \cdots$ *satisfy the following identities*:

$$\frac{1}{f_k} = \frac{1}{1+x} + (1 - g_1^{(k)}) \frac{x}{1+x} \frac{1}{f_{k+1}},$$

$k = 0, 1, 2, \cdots,$ *and consequently* $1/f(x)$ *has the Euler expansion*

$$\frac{1}{f(x)} = \frac{1}{1+x} + (1 - g_1) \frac{x}{(1+x)^2} + (1 - g_1)(1 - g_1^{(1)}) \frac{x^2}{(1+x)^3} + \cdots.$$

COROLLARY 5.1. *Let* $c_0 \neq 0$, *and* $c_0/f(x) = c_0 - c_1 x + c_2 x^2 - \cdots$. *Then*

$$(5.4) \qquad \Delta^m c_0 = c_0(1 - g_1)(1 - g_1^{(1)}) \cdots (1 - g_1^{(m-1)}).$$

COROLLARY 5.2. *The correspondence*

$$(5.5) \quad \Delta^m c_0 - \Delta^m c_1 x + \Delta^m c_2 x^2 - \cdots \sim \frac{\Delta^m c_0}{1} + \frac{g_1^{(m)} x}{1} + \frac{(1 - g_1^{(m)})g_2^{(m)} x}{1} + \frac{(1 - g_2^{(m)})g_3^{(m)} x}{1} + \cdots$$

is valid for $m = 0, 1, 2, \cdots$.

From (5.4) it follows that if $c_0 > 0$ then $\Delta^m c_0 > 0$ for $m = 1, 2, 3, \cdots$. We shall show next that $\Delta^m c_n > 0$ for all m, n. To do this it will suffice to prove

THEOREM 5.2. *If* $0 < g_n < 1$ *and*

$$c_0 - c_1 x + c_2 x^2 - \cdots \sim c_0/1 + g_1 x/1 + (1 - g_1)g_2 x/1 + (1 - g_2)g_3 x/1 + \cdots,$$

then

$$c_1 - c_2 x + c_3 x^2 - \cdots \sim c_1/1 + h_1 x/1 + (1 - h_1) h_2 x/1$$
$$+ (1 - h_2) h_3 x/1 + \cdots ,$$

where $0 < h_n < 1$.

Proof. With the aid of Stieltjes integrals and the results of §4, one may prove this theorem in a few lines. The proof may be made by means of the continued fraction algorithm alone by means of the four lemmas which follow.

Lemma 5.1. *The continued fraction*[10]

$$u_1 + v_1 - \cfrac{u_1 v_2}{u_2 + v_2} - \cfrac{u_2 v_3}{u_3 + v_3} - \cdots - \cfrac{u_{n-1} v_n}{u_n + v_n}$$

in which $u_i \neq 0$, $i = 1, 2, 3, \cdots, n$, *is equal to*

$$v_1 + \cfrac{u_1}{1 + \cfrac{v_2}{u_2} + \cfrac{v_2 v_3}{u_2 u_3} + \cdots + \cfrac{v_2 v_3 \cdots v_n}{u_2 u_3 \cdots u_n}} .$$

This may be readily proved by mathematical induction.

Lemma 5.2. *If* $a_n > 0$ *and*

$$c_0 - c_1 x + c_2 x^2 - \cdots \sim a_1/1 + a_2 x/1 + a_3 x/1 + \cdots ,$$

then[11]

$$c_1 - c_2 x + c_3 x^2 - \cdots \sim b_1/1 + b_2 x/1 + b_3 x/1 + \cdots ,$$

where

$$(5.6) \qquad b_1 = a_1 a_2, \qquad b_2 = a_2 + a_3, \qquad b_{2n+1} = a_{2n+3} + a_{2n+2} - b_{2n+2},$$

$$(5.7) \qquad b_{2n+2} = a_{2n+3} + \cfrac{a_{2n+2}}{1 + \cfrac{a_{2n+1}}{a_{2n}} + \cfrac{a_{2n+1} a_{2n-1}}{a_{2n} a_{2n-2}} + \cdots + \cfrac{a_{2n+1} a_{2n-1} \cdots a_3}{a_{2n} a_{2n-2} \cdots a_2}} .$$

Proof. It is well known that $c_1 - c_2 x + c_3 x^2 - \cdots$ has a corresponding continued fraction of the form specified. Moreover, the "odd part" of $a_1/1 + a_2 x/1 + a_3 x/1 + \cdots$ must be the same as the "even part" of $b_1/1 + b_2 x/1 + b_3 x/1 + \cdots$. That is,

$$(5.8) \quad \begin{aligned} & a_1 - \cfrac{a_1 a_2 x}{1 + (a_2 + a_3) x} - \cfrac{a_3 a_4 x^2}{1 + (a_4 + a_5) x} - \cfrac{a_5 a_6 x^2}{1 + (a_6 + a_7) x} - \cdots \\ & = c_0 - \cfrac{b_1 x}{1 + b_2 x} - \cfrac{b_2 b_3 x^2}{1 + (b_3 + b_4) x} - \cfrac{b_4 b_5 x^2}{1 + (b_5 + b_6) x} - \cdots . \end{aligned}$$

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[10] O. Szász, *Ueber die Erhaltung der Konvergenz unendlicher Kettenbrüche* $\cdots$, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 132–160.

[11] This holds for any a_n's such that there is no division by zero in the formulas.

On equating corresponding elements in these continued fractions we obtain (5.6) and the relation $b_{2n}b_{2n+1} = a_{2n+1}a_{2n+2}$. Hence, on combining this with (5.6),

$$b_{2n+2} = a_{2n+3} + a_{2n+2} - \frac{a_{2n+1}a_{2n+2}}{a_{2n+1}+a_{2n}} - \frac{a_{2n-1}a_{2n}}{a_{2n-1}+a_{2n-2}} - \cdots - \frac{a_3a_4}{a_2+a_3},$$

which, by Lemma 5.1, reduces to (5.7).

LEMMA 5.3. *If in Lemma 5.2 the a_n's have the form $a_2 = g_1$, $a_n = g_{n-1}(1-g_{n-2})$, $n > 2$, where $0 < g_n < 1$, then $0 < b_n < 1$, $n \geq 2$.*

Proof. We have $0 < b_2 = 1 - (1-g_1)(1-g_2) < 1$, $0 < b_{2n+2} = 1 - (1-g_{2n+1})$ $\cdot (1-g_{2n+2}) - g_{2n+1}g_{2n} < 1$. Hence also, $0 < b_{2n+1} < 1$.

LEMMA 5.4. *Under the hypothesis of Lemma 5.3 the numerators of the approximants of the continued fraction $1 + b_2x/1 + b_3x/1 + b_4x/1 + \cdots$ are all positive for $x = -1$.*

Proof. Let $A_n(x)$, $n = 0, 1, 2, \cdots$, be the numerators in question, and let $C_n(x)$ be the denominator of the nth approximant of

$$(5.9) \qquad a_1/1 + a_2x/1 + a_3x/1 + \cdots.$$

Then by (5.8), $C_{2n+1}(x) \equiv A_{2n}(x)$. Now $\left| C_2(x) \right| = \left| 1 + g_1x \right| \geq (1-g_1) \left| C_1(x) \right|$, if $\left| x \right| \leq 1$. If, for any value of n, $\left| x \right| \leq 1$ implies that

$$\left| C_n(x) \right| \geq (1 - g_{n-1}) \left| C_{n-1}(x) \right| \geq (1 - g_1)(1 - g_2) \cdots (1 - g_{n-1}),$$

then $\left| C_{n+1}(x) \right| = \left| C_n(x) + g_n(1-g_{n-1})xC_{n-1}(x) \right| \geq \left| C_n(x) \right| - g_n(1-g_{n-1}) \left| C_{n-1}(x) \right|$ if $\left| x \right| \leq 1$. Hence[12]

$$\left| C_{n+1}(x) \right| \geq (1 - g_n) \left| C_n(x) \right| \geq (1 - g_1)(1 - g_2) \cdots (1 - g_n).$$

Inasmuch as $C_n(0) = 1$, we must therefore have $C_{2n+1}(-1) = A_{2n}(-1) > 0$. Now $A_{2n+2}(x) = A_{2n+1}(x) + b_{2n+3}xA_{2n}(x)$. Hence if $A_{2n+1}(x) = 0$ for $-1 \leq x \leq 0$, then $A_{2n+2}(x)$ and $A_{2n}(x)$ would have opposite signs for this value of x. Since this is impossible, it follows that $A_{2n+1}(-1) > 0$.

The proof of Theorem 5.2 may now be readily made. By Lemma 5.4 we may apply Theorem 2.3 with $c = 1$ and obtain $b_2 = h_1$, $b_n = h_{n-1}(1-h_{n-2})$, $n > 2$, where $h_n = b_{n+1}A_{n-2}(-1)/A_{n-1}(-1) > 0$. But by Lemma 5.3, $0 < h_1 < 1$, $0 < h_{n-1}(1-h_{n-2}) < 1$, $m > 2$. Consequently $0 < h_n < 1$, $n \geq 1$.

THEOREM 5.3. *If $c_0 > 0$, and*

$$c_0 - c_1x + c_2x^2 - \cdots \sim c_0/1 + g_1x/1 + (1 - g_1)g_2x/1$$
$$+ (1 - g_2)g_3x/1 + \cdots,$$

where $0 < g_n < 1$, then

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[12] This induction was used by Van Vleck, loc. cit.

$$(5.10) \qquad\qquad \Delta^m c_n > 0, \qquad\qquad m, n = 0, 1, 2, \cdots,$$

and

$$(5.11) \qquad a < \frac{\Delta c_n}{c_n} < \frac{\Delta^2 c_n}{\Delta c_n} < \frac{\Delta^3 c_n}{\Delta^2 c_n} < \cdots < 1, \qquad n = 0, 1, 2, \cdots.$$

Proof. The inequalities (5.10) follow from (5.4) and Theorem 5.2. It will suffice to prove (5.11) for the case $n = 0$. By (5.4) we have

$$1 > \frac{\Delta^{m+1} c_0}{\Delta^m c_0} = (1 - g_1^{(m)}) = 1 - g_1^{(m-1)}(1 - g_2^{(m-1)})$$

$$> 1 - g_1^{(m-1)} - \frac{\Delta^m c_0}{\Delta^{m-1} c_0} > 0,$$

$m = 1, 2, 3, \cdots$, which was to be proved.

6. **Totally monotone sequences corresponding to a finite distribution of mass.** Consider a terminating continued fraction of the form $c_0/1 + g_1 x/1 + (1 - g_1)g_2 x/1 + \cdots + (1 - g_{k-1})g_k x/1$, where $c_0 > 0$, $0 < g_n < 1$, $n = 1, 2, 3, \cdots$, $k - 1$, $0 < g_k \leq 1$. We know that if $k = 2m - 1$ this continued fraction represents a rational function of the form

$$\frac{A_{2m}(x)}{B_{2m}(x)} = \frac{M_1}{1 + xx_1} + \frac{M_2}{1 + xx_2} + \cdots + \frac{M_m}{1 + xx_m},$$

where $M_i > 0$, $i = 1, 2, 3, \cdots, m$, and $0 < x_1 < x_2 < \cdots < x_m \leq 1$. On the other hand, if $k = 2m$, the function represented has the form

$$\frac{A_{2m+1}(x)}{B_{2m+1}(x)} = M_0 + \frac{M_1}{1 + xx_1} + \frac{M_2}{1 + xx_2} + \cdots + \frac{M_m}{1 + xx_m},$$

where $M_i > 0$, $i = 0, 1, 2, \cdots, m$, and $0 < x_1 < x_2 < \cdots < x_m \leq 1$. Naturally the M_i's, x_i's are not the same in the two cases. In either case if $c_0 - c_1 x + c_2 x^2 - \cdots$ is the corresponding power series, then

$$c_n = \sum_{i=0}^{i=m} x_i^n M_i, \qquad n = 0, 1, 2, \cdots,$$

where $x_0 = 0$, $x_0^0 = 1$, and M_0 is positive or 0 according as $k = 2m$ or $2m - 1$. Thus we may write $c_n = \int_0^1 u^n d\phi(u)$, $n = 0, 1, 2, \cdots$, where $\phi(u)$ is a monotone nondecreasing function with but a finite number of points of increase.

Conversely, let $c_0, c_1, c_2, \cdots$ be a totally monotone sequence corresponding to a finite distribution of mass. Then c_n has a representation of the form

$$(6.1) \qquad\qquad c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots,$$

where $\phi(u)$ is a step-function. There are two cases to be considered according as $\phi(u)$ is or is not continuous at $u = 0$.

LEMMA 6.1. *Let* A_n, B_n *denote the determinants* $|c_{i+j}|$, $|c_{i+j+1}|$ $(i, j = 0, 1, \cdots, n)$, *respectively. Then if* $\phi(u)$ *is continuous at* $u = 0$, *these determinants are positive for* $n < m$, *and if* $\phi(u)$ *is discontinuous at* $u = 0$, $A_n > 0$ *for* $n \leq m$, *and* $B_n > 0$ *for* $n < m$, *where m is the number of values of* $u > 0$ *where* $\phi(u)$ *is discontinuous.*

Proof. Put

$$c_n = \sum_{i=0}^{i=m} x_i^n M_i, \qquad n = 0, 1, 2, \cdots,$$

where $0 < x_1 < x_2 < \cdots < x_m \leq 1$, M_1, M_2, $\cdots$, M_m are positive, and $x_0 = 0$, $x_0^0 = 1$. If $\phi(u)$ is continuous at $u = 0$, then $M_0 = 0$, while if $\phi(u)$ is discontinuous at $u = 0$, $M_0 > 0$. Consider the quadratic form

$$\sum_{i,j=0}^{n} c_{i+j+r} \xi_i \xi_j = \int_0^1 u^r \left\{ \sum_{i=0}^{i=n} \xi_i u^i \right\}^2 d\phi(u).$$

If $r = 1$ this is clearly positive definite for $n < m$, so that $B_n > 0$ for $n < m$ in both cases. When $r = 0$, it is positive definite for $n < m$ if $\phi(u)$ is continuous at $u = 0$, and for $n \leq m$ if $\phi(u)$ is discontinuous at $u = 0$. Hence $A_n > 0$ in the first case for $n < m$, and in the second for $n \leq m$.

Now

$$P(x) \equiv c_0 - c_1 x + c_2 x^2 - \cdots \equiv M_0 + \frac{M_1}{1 + x x_1} + \frac{M_2}{1 + x x_2} + \cdots$$
$$+ \frac{M_m}{1 + x x_m} = \frac{S(x)}{T(x)},$$

where $T(x)$ is a polynomial of degree m, and $S(x)$ is of degree m or $m - 1$ according as $M_0 > 0$ or $M_0 = 0$. Put

$$a_1 = c_0, \qquad a_{2n} = B_{n-1} A_{n-2} / A_{n-1} B_{n-2}, \qquad a_{2n+1} = A_n B_{n-2} / B_{n-1} A_{n-1},$$

$n = 1, 2, 3, \cdots$, $A_{-1} = B_{-1} = 1$, and form the continued fractions

$$(6.2) \qquad \frac{A_{2m}(x)}{B_{2m}(x)} = \frac{a_1}{1} + \frac{a_2 x}{1} + \cdots + \frac{a_{2m} x}{1} + \frac{0 \cdot x}{1},$$

and

$$(6.3) \qquad \frac{A_{2m+1}(x)}{B_{2m+1}(x)} = \frac{a_1}{1} + \frac{a_2 x}{1} + \cdots + \frac{a_{2m+1} x}{1} + \frac{0 \cdot x}{1},$$

according as $M_0 = 0$ or $M_0 > 0$. The vanishing partial quotient is affixed for a

reason to appear presently. On taking account of the degrees of numerators and denominators, and of the degree of approximation of these rational fractions to the power series $P(x)$, we conclude that they are identical with $S(x)/T(x)$ in the two cases.

The denominators of the nth approximants of (6.2), (6.3) are greater than 0 if $-1 \leqq x \leqq 0$, and $n < 2m$, $n < 2m+1$, respectively. Hence we may apply Theorem 2.3 with $c = 1$ to show that the a_n's have the form $g_{n-1}(1 - g_{n-2})$, $n > 2$, $a_2 = g_1$. We then apply Theorem 2.2 and obtain in the case of (6.2):

$$\Delta c_0 - \Delta c_1 x + \Delta c_2 x^2 - \cdots \sim \frac{\Delta c_0}{1} + \frac{g_1(1 - g_1)x}{1} + \cdots$$
$$+ \frac{g_{2m-2}(1 - g_{2m-1})x}{1} + \frac{g_{2m-1}x}{1}.$$

It is easy to see that $\phi(u)$ is discontinuous at $u = 1$ if and only if $g_{2m-1} = 1$. In case $g_{2m-1} = 1$, the above continued fraction terminates with the $(2m-2)$th partial quotient, while if $g_{2m-1} < 1$ it terminates with the $2m$th partial quotient.

Now $\Delta c_n = \int_0^1 u^n d\phi_1(u)$, where $\phi_1(u) = \int_0^u (1 - u)d\phi(u)$. Hence $\phi_1(u)$ has the same number of discontinuities as, or a smaller number by one than, $\phi(u)$, according as 1 is not or is, respectively, a point of discontinuity of $\phi(u)$. Since $\phi_1(u)$ is a function of the same character as $\phi(u)$, we conclude that $1 - g_1 > 0$, $g_{n-1}(1 - g_n) > 0$, $n = 2, 3, \cdots, 2m-2$ if $g_{2m-1} = 1$, and $n = 2, 3, \cdots, 2m-1$ if $g_{2m-1} < 1$. But we previously had $g_1 > 0$, $g_n(1 - g_{n-1}) > 0$, $n = 2, 3, \cdots, 2m-1$. Hence we conclude that $0 < g_n < 1$, $n = 1, 2, 3, \cdots, 2m-2$, $0 < g_{2m-1} \leqq 1$. The treatment of (6.3) is exactly the same. We have therefore completed the proof of the following theorem:

THEOREM 6.1. *If* $c_0, c_1, c_2, \cdots$ *is a totally monotone sequence corresponding to a finite distribution of mass, then* $c_0 - c_1 x + c_2 x^2 - \cdots$ *is the constant* $c_0 \geqq 0$, *or else*

$$c_0 - c_1 x + c_2 x^2 - \cdots \sim c_0/1 + g_1 x/1 + (1 - g_1)g_2 x/1 + \cdots$$
$$+ g_k(1 - g_{k-1})x/1,$$

where $0 < g_n < 1$, $n < k$, $0 < g_k \leqq 1$, $c_0 > 0$. *Conversely, any sequence determined in this way is a totally monotone sequence corresponding to a finite distribution of mass.*

7. **The moment problem for the interval** $(-\infty, 1)$. The methods used previously may be employed to prove the following theorem:

THEOREM 7.1. *If* $\phi(u)$ *is a monotone nondecreasing function in the interval* $(-\infty, 1)$, *such that the moments* $c_n = \int_{-\infty}^1 u^n d\phi(u)$, $n = 0, 1, 2, \cdots$, *are all finite, and if the series* $c_0 - c_1 x + c_2 x^2 - \cdots$ *has a corresponding continued fraction,*

then this continued fraction must have the form

(7.1) $c_0/1 + g_1x/1 + (1 - g_1)g_2x/1 + (1 - g_2)g_3x/1 + \cdots$,

where $c_0 > 0$, and

$$g_{2n-1}g_{2n} > 0, \qquad (1 - g_{2n-1})(1 - g_{2n-2}) > 0,$$

$n = 1, 2, 3, \cdots, g_0 = 0$.

Proof. Let $b_1/1 + b_2x/1 + b_3x/1 + \cdots$ be the corresponding continued fraction which is supposed to exist. Then it is known[13] that $b_1 > 0$, $b_{2n}b_{2n+1} > 0$, $n = 1, 2, 3, \cdots$. The zeros of the denominators $B_n(x)$ of the approximants of this continued fraction are all real and none of them lie in the interval[14] $-1 \leq x \leq 0$. Hence $B_n(-1) > 0$. By Theorem 2.3 we may then write this continued fraction in the form (7.1) where $g_n \neq 0, 1, n \geq 1$. Then by Theorem 2.2 we have

$$\Delta c_0 - \Delta c_1 x + \Delta c_2 x^2 - \cdots \sim \frac{\Delta c_0}{1 +} \quad \frac{g_1(1 - g_2)x}{1 +} \quad \frac{g_2(1 - g_3)x}{1 +} \cdots.$$

Now $\Delta^m c_n = \int_{-\infty}^1 (1-u)^m u^n d\phi(u) = \int_{-\infty}^1 u^n d\phi_1(u)$, where $\phi_1(u) = \int_{-\infty}^u (1-u)^m d\phi(u)$ is monotone nondecreasing for $-\infty < u \leq 1$, and consequently $\Delta c_0 = c_0(1-g_1) > 0$, $g_{2n-1}g_{2n}(1-g_{2n})(1-g_{2n+1}) > 0$, $n \geq 1$. The theorem now follows. We shall leave unanswered the question of the converse of this theorem.

8. **Moment generating functions which are bounded in the unit circle.** The problem of this section is to specialize the continued fractions of Theorems 4.1 and 6.1 in such a way that the *moment generating function* $f(x) = c_0 - c_1x + c_2x^2 - \cdots$ will be bounded in the unit circle, that is

$$M(f) = \underset{|x|<1}{\text{l.u.b.}} \; |f(x)|$$

will be finite. Our first result is contained in the theorem which follows.

THEOREM 8.1. *The function $f(x) = c_0 - c_1x + c_2x^2 - \cdots$ is in the class E, of moment generating functions bounded in the unit circle, if and only if there is a correspondence of the form*

$$hf(x) \sim g_1/1 + (1 - g_1)g_2x/1 + (1 - g_2)g_3x/1 + \cdots ,$$

for some sufficiently small positive number h and for real g_n's such that $0 \leq g_n \leq 1$, $n = 1, 2, 3, \cdots$, where it is agreed that the continued fraction shall terminate with the first identically vanishing partial quotient. When the condition is satisfied, $M(f) \leq 1/h$, and h can be taken equal to 1 if and only if $M(f) \leq 1$.

Proof of sufficiency. Suppose first that the continued fraction terminates

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[13] H. Hamburger, Mathematische Annalen, vol. 81 (1920), pp. 235–319, and vol. 82 (1921), pp. 120–187.
[14] Cf. footnote 8.

and that $hf(x) = g_1$, $0 \leq g_1 \leq 1$, or $hf(x) = g_1/1 + (1-g_1)g_2x/1 + \cdots + (1-g_k)$ $\cdot g_{k+1}x/1$, $k \geq 1$, $0 < g_n < 1$, $n = 1, 2, 3, \cdots, k$, $0 < g_{k+1} \leq 1$. The first possibility is at once disposed of. When $g_{k+1} < 1$, the second possibility is disposed of by Theorem 3.1; and when $g_{k+1} = 1$, it is required to be shown that the rational function $hf(x)$ has no poles for $|x| \leq 1$. To do this, recall that the function $1/(1+xhf(x))$ has the form

$$\frac{M_0}{1+xx_0} + \frac{M_1}{1+xx_1} + \cdots + \frac{M_{m-1}}{1+xx_{m-1}} + \frac{M_m}{1+xx_m}$$

where $x_0 = 0 < x_1 < x_2 < \cdots < x_m \leq 1$, $M_0 \geq 0$, $M_i > 0$, $i = 1, 2, \cdots, m$, and that in the case under consideration $x_m = 1$. Consequently, the zero of this function which lies nearest the origin is less than -1, so that the function $1+xhf(x)$ has no pole for $|x| \leq 1$ and therefore $M(f)$ is finite. It will be seen that $f(-1) = 1/h = c_0 + c_1 + c_2 + \cdots = M(f)$.

When the continued fraction does not terminate, then by Theorem 3.1, $M(f) \leq 1/h$, and $f(x) \, \varepsilon \, E$.

Proof of necessity. Suppose conversely that $f(x) \, \varepsilon \, E$, and let $f(x) = c_0/1 + h_1x/1 + (1-h_1)h_2x/1 + (1-h_2)h_3x/1 + \cdots$, where $c_0 \geq 0$, $0 \leq h_n \leq 1$, $n \geq 1$. If $c_0 = 0$ or $h_1 = 0$, the theorem is obviously true. Suppose $c_0 > 0$, $0 < h_n < 1$, $n = 1, 2, \cdots, k-1$, $0 < h_k \leq 1$, and that the continued fraction terminates by having $h_{k+1}(1-h_k) = 0$. Then we must have $h_k < 1$, for otherwise $f(x)$ has a pole at $x = -1$. Now consider, whether or not the continued fraction terminates, the function

$$1/(1 + xhf(x)) = 1/1 + hc_0x/1 + h_1x/1 + (1 - h_1)h_2x/1$$
$$+ (1 - h_2)h_3x/1 + \cdots.$$

Since $0 < h_1 < 1$, $0 \leq h_n < 1$, $n > 1$, the continued fraction $1 + h_1x/1 + (1-h_1)h_2x/1 + (1-h_2)h_3x/1 + \cdots$ converges uniformly for $|x| \leq 1$, by Theorem 3.1, and cannot vanish for $|x| \leq 1$ since $M(f) < \infty$ by hypothesis. It readily follows that $h > 0$ can be so chosen ($h = 1$ if $M(f) \leq 1$), that for every r, $0 < r < 1$, the continued fraction for $1/(1+xhf(x))$ converges uniformly for $|x| \leq r$, and consequently we must have, for the chosen value of h,

$$1/(1 + xhf(x)) = \int_0^1 d\phi(u)/(1 + xu),$$

where $\phi(u)$ is some monotone nondecreasing function. Hence $1/(1+xhf(x))$ is a moment generating function, and therefore has a representation of the form $1/1 + g_1x/1 + (1-g_1)g_2x/1 + (1-g_2)g_3x/1 + \cdots$, so that $hf(x)$ has a representation as prescribed by the theorem.

Another characterization of E in terms of continued fractions is given by the theorem which follows:

THEOREM 8.2. *Given a moment generating function* $f(x) = c_0/1 + g_1 x/1 + (1-g_1)g_2 x/1 + \cdots$ $(c_0 \geqq 0,\ 0 \leqq g_n \leqq 1)$, $f(x)\ \varepsilon\ E$ *if and only if* $f(x) \equiv c_0 \geqq 0$, *or* $f(x) = c_0/1 + g_1 x/1 + (1-g_1)g_2 x/1 + \cdots + (1-g_{k-2})g_{k-1}x/1$, *where* $c_0 > 0$, $0 < g_n < 1$, $n = 1,\ 2,\ 3,\ \cdots,\ k-1$; *or else* $f(x) = c_0/1 + g_1 x/1 + (1-g_1)g_2 x/1 + (1-g_2)g_3 x/1 + \cdots$, *where* $c_0 > 0$, $0 < g_n < 1$, $n \geqq 1$, *and the series*

$$(8.1) \qquad 1 + \sum_{n=1}^{n=\infty} \frac{g_1 g_2 \cdots g_n}{(1-g_1)(1-g_2)\cdots(1-g_n)}$$

converges.

Proof. The case of the terminating continued fraction is easily disposed of, for when the condition is satisfied the rational function has no poles for $|x| \leqq 1$, while if $g_{k-1} = 1$ (the only alternative) there is a pole at $x = -1$.

In the case of the nonterminating continued fraction, the function $c_0/f(x) = 1 + g_1 x/1 + (1-g_1)g_2 x/1 + (1-g_2)g_3 x/1 + \cdots$ is analytic for $|x| < 1$ and continuous for $|x| \leqq 1$. Also by Theorem 2.1 the function $(1+x)f(x)$ enjoys these same properties. Thus $f(x)$ is continuous for $|x| \leqq 1$ except possibly at $x = -1$. But by Theorem 3.1 $f(-1)/c_0$ is equal to the series (8.1) and is therefore finite if and only if the latter converges.

9. A characterization of E in terms of the moments c_n. We now prove

THEOREM 9.1. *A moment generating function* $f(x) = c_0 - c_1 x + c_2 x^2 - \cdots$ *is in E if and only if the series* $c_0 + c_1 + c_2 + \cdots$ *converges, and when the condition is satisfied we have*

$$M(f) = c_0 + c_1 + c_2 + \cdots .$$

Proof. *Sufficiency*: When the series $c_0 + c_1 + c_2 + \cdots$ converges, the sequence $d_n = c_n + c_{n+1} + c_{n+2} + \cdots$, $n = 0, 1, 2, \cdots$, is totally monotone, so that the power series $d_0 - d_1 x + d_2 x^2 - \cdots$ has a corresponding continued fraction of the form $d_0/1 + h_1 x/1 + (1-h_1)h_2 x/1 + (1-h_2)h_3 x/1 + \cdots$, $(d_0 \geqq 0,\ 0 \leqq h_n \leqq 1)$. Then, since $\Delta d_n = c_n$, it follows from Theorem 2.2 that $c_0 - c_1 x + c_2 x^2 - \cdots \sim c_0/1 + (1-g_1)g_2 x/1 + (1-g_2)g_3 x/1 + \cdots$ where we have put $g_n = 1 - h_n$, $n = 1, 2, 3, \cdots$. That $f(x)\ \varepsilon\ E$ now follows from Theorem 8.1.

Necessity: The necessity of the condition results at once from a theorem of Abel. However, it is interesting to give a direct proof. Suppose then that $f(x)\ \varepsilon\ E$. Then by Theorem 8.1 we must have $hf(x) \sim g_1/1 + (1-g_1)g_2 x/1 + (1-g_2)g_3 x/1 + \cdots$ for some $h > 0$, where the g_n's are restricted as in Theorem 8.1. Therefore $1 + hxf(x) \sim 1 + g_1 x + (1-g_1)g_2 x/1 + (1-g_2)g_3 x/1 + \cdots$, so that, by Theorem 2.1,

$$\frac{1 + hxf(x)}{1 + x} \sim \frac{1}{1+} \frac{(1-g_1)x}{1+} \frac{g_1(1-g_2)x}{1+} \cdots .$$

Consequently the function $(1 + hxf(x))/(1+x)$ is a moment generating func-

tion. Its power series expansion is $1-(1-hc_0)x+(1-hc_0-hc_1)x^2-\cdots$, so that $1-h(c_0+c_1+c_2+\cdots+c_n)\geqq 0$ for $n=0,1,2,\cdots$. It follows that the series $\sum c_i$ is convergent.

Since $f(-1)=c_0+c_1+c_2+\cdots$, it follows that $M(f)$ is equal to this sum. As a corollary to Theorem 9.1 we have

THEOREM 9.2. *The moment generating function $f(x)$ is in E if and only if it has a Stieltjes integral representation of the form*

$$f(x) = \int_0^1 \frac{(1-u)d\phi(u)}{1+xu},$$

in which $\phi(u)$ is bounded and monotone nondecreasing.

Proof. If $f(x)\ \varepsilon\ E$, then the series $\sum c_i$ converges. Let d_n be defined as in the proof of Theorem 9.1. Then the function $d_0-d_1x+d_2x^2-\cdots$ has a representation of the form $\int_0^1 d\phi(u)/(1+xu)$, where $\phi(u)$ is bounded and monotone nondecreasing. Since we must have $f(x)=\int_0^1 (1-u)d\phi(u)/(1+xu)$, the necessity of the condition is proved. Conversely, if the condition is fulfilled, put $d_0-d_1x+d_2x^2-\cdots=\int_0^1 d\phi(u)/(1+xu)$, where $\phi(u)$ is given, Then $c_n=\Delta d_n$, so that $\sum c_i=d_0-\lim_{n=\infty}d_n$ is convergent, and consequently $f(x)\ \varepsilon\ E$.

10. The algorithm of Schur. Starting with a function $f(x)$ of E for which $M(f)\leqq 1$, we construct with Schur[15] a sequence of functions $f_n(x)$ in the following way. Put $f_0(x)=f(x)$,

$$(10.1) \qquad f_{n+1}(x) = \frac{1}{x}\ \frac{t_n-f_n(x)}{1-t_nf_n(x)}, \qquad t_n = f_n(0), \qquad n = 0, 1, 2, \cdots .$$

If $M(f)=1$, then $M(f_n)=1$, and if $M(f)<1$, then $M(f_n)<1$, $(n=1,2,3,\cdots)$.

If $f(x)\equiv g_1$, then $f_n(x)\equiv 0$ for $n=1,2,3,\cdots$. If $f(x)=g_1/1+(1-g_1)g_2x/1$, then $f_1(x)=g_1^{(1)}/1+(1-g_1^{(1)})g_2^{(1)}x/1$, where

$$g_1^{(1)} = g_1g_2/(1+g_1), \qquad g_2^{(1)} = g_1g_2/(1+g_1-g_1g_2),$$

so that if $0<g_1<1$, $0<g_2\leqq 1$, then $0<g_1^{(1)}<1$, $0<g_2^{(1)}\leqq 1$. We shall prove this theorem:

THEOREM 10.1. *If $f(x)=g_1/1+(1-g_1)g_2x/1+(1-g_2)g_3x/1+\cdots$, $0\leqq g_n\leqq 1$, $n=1,2,3,\cdots$, with the agreement that the continued fraction shall terminate with the first identically vanishing partial numerator, so that $f(x)$ is a moment generating function and $M(f)\leqq 1$, then the functions $f_1(x),f_2(x),f_3(x),\cdots$ given by (10.1) are all moment generating functions and $M(f_n)\leqq 1$. The only case where any of these functions are constants is where $f(x)\equiv g_1$, whereupon $f_n(x)\equiv 0$, $n=1,2,3,\cdots$.*

Proof. By (10.1), if $f(x)$ is not a constant, then

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[15] Cf. footnote 4.

$$(10.2) \quad xf_1(x) = g_1 - \frac{(1 - g_1^2)}{- g_1 + 1/f(x)}$$

$$= g_1 - g_1/1 + g_2(1 - h_1)x/1 + g_3(1 - g_2)x/1 + \cdots ,$$

where we have put $1 - h_1 = 1/(1 + g_1)$, $h_1 = g_1/(1 + g_1)$. This function is evidently of the form $g_1 - g(x)$, where $g(x)$ is a moment generating function and $g(0) = g_1$. Put $g(x) = g_1 - d_0 x + d_1 x^2 - \cdots$, and we find that $f_1(x) = d_0 - d_1 x + d_2 x^2 - \cdots$ and is therefore a moment generating function. On account of the character of the transformation used, $M(f_1) \le 1$. Consequently, by Theorem 8.1, $f_1(x) = g_1^{(1)}/1 + (1 - g_1^{(1)})g_2^{(1)}x/1 + (1 - g_2^{(1)})g_3^{(1)}x/1 + \cdots$ where $0 \le g_n^{(1)} \le 1$, $n = 1, 2, 3, \cdots$, with the oft repeated convention regarding termination of the continued fraction.

In the same way, starting with $f_1(x)$ instead of with $f(x)$, we find that $f_2(x)$ is a moment generating function, and $M(f_2) \le 1$; and by induction, $f_3(x), f_4(x), \cdots$ all have this property.

We saw previously that when $f(x) = g_1$, then $f_n(x) = 0$, $(n = 1, 2, 3, \cdots)$. It remains to be shown that this is the only case where any of the functions can reduce to a constant. To do this, it suffices to show that if $f_1(x) \equiv c$, a constant, then $c = 0$. We have

$$f(x) = \frac{g_1 - cx}{1 - cxg_1} = g_1 - (1 - g_1^2)cx - g_1 c^2(1 - g_1^2)x^2 - \cdots .$$

Since this is a moment generating function, we must have $-g_1 c^2(1 - g_1^2) \ge 0$, which implies that $c = 0$, or else $g_1 = 0$ or 1. But when $g_1 = 0$ or 1, we must have $f = g_1$, and consequently $f_1 = c = 0$ in this case also.

As a corollary we have

THEOREM 10.2. *If $f(x)$ is a moment generating function for which $M(f) \le 1$, then the sequence $t_0, t_1, t_2, \cdots$ given by* (10.1) *has the property of a totally monotone sequence that if any member is* 0 *the others are also* 0 *with the possible exception of the first.*

In a number of examples which we have examined, the sequence $\{t_n\}$ has been found to be totally monotone. Technical difficulties have thus far prevented us from determining whether or not this is always the case. Also, the question as to the converse of this naturally arises, namely, if $\{t_n\}$ is a totally monotone sequence, will the function $f(x)$ which this sequence determines be a moment generating function with $M(f) \le 1$?

In conclusion we shall give recursion formulas for computing the t_n's in terms of the g_n's. From (10.2) we have

$$xf_1(x) = g_1 - \frac{g_1}{1 +} \frac{(1 - h_1)g_2 x}{1 +} \frac{(1 - g_2)g_3 x}{1 +} \cdots ,$$

which is equal to $g_1^{(1)}x/1 + (1 - g_1^{(1)})g_2^{(1)}x/1 + (1 - g_2^{(1)})g_3^{(1)}x/1 + \cdots$. On equating the odd part of the first of these continued fractions to the even part of the second we obtain at once the formulas

$$g_1^{(1)} = g_1 g_2/(1 + g_1), \qquad g_2^{(1)}(1 - g_1^{(1)}) = \frac{g_2}{1 + g_1} + g_3(1 - g_2),$$

(10.3)
$$g_n^{(1)} g_{n+1}^{(1)}(1 - g_{n-1}^{(1)})(1 - g_n^{(1)}) = g_{n+1}g_{n+2}(1 - g_n)(1 - g_{n+1}),$$

$$g_{n+1}^{(1)}(1 - g_n^{(1)}) + g_{n+2}^{(1)}(1 - g_{n+1}^{(1)}) = g_{n+2}(1 - g_{n+1}) + g_{n+3}(1 - g_{n+2}),$$

$n = 2, 4, 6, \cdots$. By means of these relations one may show that the sequence $t_n = g_1^{(n)}$, $n = 0, 1, 2, \cdots$ ($g_1^{(0)} = g_1$) is monotone decreasing. In fact, $t_n < t_{n-1}/(1 + t_{n-1})$, $n = 1, 2, 3, \cdots$.

Using a result of Schur[16] one may obtain formulas for the t_n's in terms of the g_n's. To do this, let the nth approximant of the continued fraction for $f(x)$ be

$$\frac{A_n(x)}{B_n(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k}{b_0 + b_1 x + b_2 x^2 + \cdots + b_r x^r}.$$

This rational function is a moment generating function, and its modulus is less than or equal to 1 for $|x| \leq 1$ provided $M(f) \leq 1$. Moreover, the sequence $\{t_i\}$ for this function will be in agreement with that for $f(x)$ up to and including the term of index $n - 1$. Moreover, the a_i's and b_i's can be computed in terms of the g_i's. Using the notation of Schur, we then put

$$\delta_m = \begin{vmatrix} b_0 & 0 & \cdots 0 & a_0 & a_1 \cdots a_{m-2} & a_{m-1} \\ b_1 & b_0 & \cdots 0 & 0 & a_0 \cdots a_{m-3} & a_{m-2} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ b_{m-1} & b_{m-2} & \cdots b_0 & 0 & 0 \cdots 0 & a_0 \\ a_0 & 0 & \cdots 0 & b_0 & b_1 \cdots b_{m-2} & b_{m-1} \\ a_1 & a_0 & \cdots 0 & 0 & b_0 \cdots b_{m-3} & b_{m-2} \\ \cdot & \cdot & \cdots & \cdot & \cdots & \cdot \\ a_{m-1} & a_{m-2} & \cdots a_0 & 0 & 0 \cdots 0 & b_0 \end{vmatrix}$$

where $a_i = 0$ if $i > k$ and $b_i = 0$ if $i > r$. A formula of Schur[17] then gives for t_i

$$t_i = \left\{ 1 - \frac{\delta_{i-1}\delta_{i+1}}{\delta_i^2} \right\}^{1/2},$$

which is valid for $i = 0, 1, 2, \cdots, n - 1$.

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[16] Schur, loc. cit. (first part), pp. 213–215.
[17] Loc. cit., p. 215, formula (12).

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